Out of Bounds? Testing for Long Run Relationships under Uncertainty Over Univariate Dynamics

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Abstract

Pesaran, Shin, and Smith 2001 (PSS) proposed a bounds procedure for testing for the existence of a long run relationship between a unit root dependent variable (y_t) and a set of weakly exogenous regressors $\boldsymbol{x}_t$ when the analyst does not know whether the independent variables are stationary, unit root, or mutually cointegrated processes. This procedure recognizes the analyst's uncertainty over the nature of the regressors but not the dependent variable. We extend their analysis to the case where the analyst is uncertain whether y_t is a stationary or unit root process. In this case, the test statistics proposed by PSS are uninformative for inference on the existence of a long run relationship between y_t and $\boldsymbol{x}_t$. We propose the LRM test statistic as an alternative. Using stochastic simulations, we demonstrate the behavior of the test statistic given uncertainty about the univariate dynamics of both y_t and $\boldsymbol{x}_t$, illustrate the bounds of the test statistic, and generate small sample and approximate asymptotic critical values for the upper and lower bound for a range of sample sizes and model specifications. We demonstrate the utility of the LRM test used in a bounds testing framework by re-examining XXX

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1 Introduction

The analysis of time series data is often motivated by the desire to test for and estimate long run relationships between some scalar process, y_t, and a set of weakly exogenous regressors, $\boldsymbol{x}_t$. Typically the analyst begins by conducting (a battery of) tests designed to characterize the dynamics of each individual time series and infer whether the series are $I(0)$ stationary processes or $I(1)$ unit roots. Pending the results in this pre-testing phase, two cases are straightforward.

In the case where the time series are all $I(1)$, unit root processes, the analyst proceeds to test for a long run cointegrating relationship between y_t and $\boldsymbol{x}_t$, generally using either the Engle-Granger two-step methodology (Engle and Granger 1987) or the single equation error correction model (ECM) test (Banerjee, Dolado and Mestre 1998; Ericsson and MacKinnon 2002). If the analyst adopts the Engle-Granger two step procedure, inference proceeds with non-standard critical values. If the analyst estimates a single equation ECM, MacKinnon critical values are used to test the null hypothesis that no cointegrating relationship exists between y_t and $\boldsymbol{x}_t$ (Ericsson and MacKinnon 2002).[1] If the analyst finds evidence for cointegration, the analyst can estimate the long run cointegrating relationship in a levels regression and retrieve the short run dynamics, including the rate of return to equilibrium, from an ECM. If there is no evidence of cointegration, the analyst concludes no long run relationship exists between y_t and $\boldsymbol{x}_t$, and inference on short run dynamics proceeds in standard fashion from a regression in first differences.

In the case where the time series are all $I(0)$, stationary processes, inference about and estimation of the long run relationship proceeds in the standard linear regression framework. The analyst may choose to estimate an autoregressive distributed lag (ADL), a generalized error correction (GECM) model, or restricted versions of either model. Standard limiting distributions are used to draw inferences from long and short run parameter estimates.

The way forward is less clear a) if we are uncertain of the univariate dynamics and, in particular, b) if the data are a mix of $I(0)$ and $I(1)$ processes. Pre-testing often produces inconclusive results and may suggest some variables are $I(0)$ while others are $I(1)$.[2] Inferences about univariate dynamics are, of course, inherently uncertain. But weak tests, short time series, and ambiguous theory often produce competing evidence as to whether series are unit roots or strongly autoregressive, stationary processes. Even when all pre-testing indicates the data are unit root processes, misclassification is a significant risk Perron and Ng (1996)

Pesaran, Shin and Smith (2001) (PSS) offer a framework for testing hypotheses about the existence of a long run cointegrating relationship between y_t and $\boldsymbol{x}_t$ when y_t is known to be a unit root, but the dynamic properties of the regressors are unknown: $\boldsymbol{x}_t$ may be stationary, unit roots, or mutually cointegrated (See also Pesaran and Smith (1998); Pesaran and Shin (1998)).[3] The authors derived the limiting distributions for the standard t- and Wald (F-) statistics used to test the significance of lagged levels in an ECM for the two polar cases in which a) all regressors are stationary and b) all regressors are unit roots. The results present critical value bounds for the null hypothesis of no long run relationship. If the computed test statistic lies above or below the bounds, inference on the null is conclusive, regardless of the underlying dynamics of the regressors, but if it lies between the bounds, the test is

[1] Dickey-Fuller critical values apply to all hypothesis tests conducted on parameters estimated as coefficients on $I(1)$ variables and standard normal critical values apply to coefficients written on mean zero stationary variables.

[2] It may also be the case that a series is fractionally integrated.

[3] In principle, fractionally integrated $\boldsymbol{x}_t$ processes are also admitted in this framework.

inconclusive as reliable inference depends on knowing the true dynamics of $\boldsymbol{x}_t$. Thus, the procedure incorporates the analyst's uncertainty over the dynamic properties of $\boldsymbol{x}_t$.

While PSS permit the analyst to be uncertain about the dynamic properties of $\boldsymbol{x}_t$, they assume the analyst knows with certainty that y_t is a unit root. This assumption is the basis for the testing framework and critical values recommended by PSS. In many cases the analyst is uncertain about the properties of the regressors *and* y_t. As we demonstrate below, uncertainty about the univariate dynamics that characterize y_t renders cointegration tests that are based on inferences on the significance of lagged y_t, such as suggested by Banerjee, Dolado and Mestre (1998) and by Pesaran, Shin and Smith (2001), uninformative for the purpose of inferring the existence of a long run relationship between y_t and $\boldsymbol{x}_t$. As PSS note, the alternative hypothesis for both tests are consistent with multiple types of long run behavior, including degenerate equilibria in which y_t is a stationary process independent of $\boldsymbol{x}_t$. The problem occurs because as y_t departs from a unit root, the coefficient on lagged levels of y_t will diverge from zero. The null of no cointegration will be rejected with increasing frequency, even if y_t is unrelated to $\boldsymbol{x}_t$ in the long run.

What should analysts do when they are uncertain whether the time series processes are $I(0)$, $I(1)$ or $I(d)$? We propose the analyst conduct inference based on the significance of the long run multiplier (LRM) relating each x_t to y_t from either the autoregressive distributed lag (ADL) or error correction model (ECM). We adopt a bounds testing framework similar to that proposed by PSS to assess the existence of a long run relationship between y_t and $\boldsymbol{x}_t$. This testing framework applies whether y_t is $I(0)$, $I(1)$, or $I(d)$ and whether the $\boldsymbol{x}_t$ are individually $I(0)$, $I(1)$, $I(d)$ or cointegrated.[4]

We begin by identifying the model and assumptions underlying our analysis.[5] We then describe the null and alternative hypotheses and test statistics underlying the PSS analysis and explicate the limitations of the tests. Specifically, we show that neither the t-test nor the Wald (F)-test statistic presented by the authors discriminates between a number of alternatives.[6] In particular, given uncertainty over the dynamic properties of y_t, the tests cannot arbitrate between simple unconditional equilibration in y_t and a long run relationship between y_t and $\boldsymbol{x}_t$. Next, we propose a test for the null hypothesis of no long run relationship between y_t and weakly exogenous $\boldsymbol{x}_t$ based on estimates of the long run multiplier (LRM) and generate critical values for the test for the upper and lower bounds to be used in a bounds testing framework applied to estimates of the t-statistics on the LRM. We end by demonstrating the utility of the test as applied in the bounds testing framework of PSS in a re-examination of XXX.

2 The Model and Assumptions

We begin by describing the basic data generating process (DGP) and assumptions underlying our analysis and that of PSS. Briefly, we begin with a vector autoregression (VAR) in which each variable in the system $\boldsymbol{z}_t$ is a function of its own lag(s), current and lagged values of all other variables in the system, and both a constant and trend. We assume the highest order of integration of any of the component variables is one and that the error in the model is well behaved. We then express the VAR as a vector error correction model (VECM), which isolates

[4]To date we have considered the cases where $\boldsymbol{x}_t$ are $I(0)$ or $I(1)$ but not the case where the $\boldsymbol{x}_t$ are cointegrated.

[5]These are the same model and assumptions underlying the analysis presented by PSS.

[6]PSS recognize the ambiguity in the alternative hypotheses but note that if y_t is known to be a unit root and we adopt a combination of tests, this ambiguity is of little concern. Given uncertainty over the univariate properties of y_t, we show that the ambiguity is problematic.

the long run relationships of interest. We assume a set of variables, $\boldsymbol{x}_t$, are weakly exogenous for the parameters in a conditional model of the scalar y_t – the variable of interest – but themselves may be $I(0)$, $I(1)$ or cointegrated. This permits hypothesis testing based on estimation of the conditional ECM. In section 3 we describe the hypothesis tests recommended by PSS and their limits, particularly when we are uncertain whether y_t is a unit root or stationary process.

The data generating process underlying what follows is a general VAR of order p (VAR(p)) for $\{\boldsymbol{z}_t\}_{t=1}^{\infty}$ a $(k+1)$-vector process. Adopting the notation in PSS, we write the model using lag operator notation as follows:

$$\boldsymbol{\Phi}(L)(\boldsymbol{z_t} - \boldsymbol{\mu} - \boldsymbol{\gamma t}) = \boldsymbol{\epsilon_t} \tag{1}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are unknown $(k+1)$-vectors of intercept and trend coefficients and $\boldsymbol{\Phi}(L)$ is a $(k+1, k+1)$ matrix lag polynomial equal to $\boldsymbol{I}_{k+1} - \sum_{i=1}^{p} \boldsymbol{\Phi}_i L^i$ with $\{\Phi_i\}_{i=1}^{p}$ $(k+1, k+1)$ matrices of unknown coefficients. All variables are at most $I(1)$ (PSS Assumption 1)[7] and the vector error process $\{\boldsymbol{\epsilon}_t\}_{t=1}^{\infty}$ is $N(\boldsymbol{0}, \boldsymbol{\Omega})$, with $\boldsymbol{\Omega}$ positive definite, allowing for contemporaneous correlations in $\boldsymbol{z}_t$ (PSS Assumption 2).

We reparameterize the VAR as a Vector Error Correction Model (VECM) to isolate the long run, levels relationship of interest among the variables. Setting $\boldsymbol{\Phi}(L) \equiv -\boldsymbol{\Pi} L + \boldsymbol{\Gamma}(L)(1-L)$ we can express the VAR as an equivalent VECM given by

$$\Delta \boldsymbol{z}_t = \boldsymbol{a}_0 + \boldsymbol{a_1} t + \boldsymbol{\Pi} \boldsymbol{z_{t-1}} + \Sigma_{i=1}^{p-1} \boldsymbol{\Gamma}_i \Delta \boldsymbol{z}_{t-i} + \boldsymbol{\epsilon}_t{}^{8} \tag{2}$$

where $\Delta \equiv 1 - L$ is the difference operator and the matrix of long run multipliers is given by $\boldsymbol{\Pi} \equiv -(\boldsymbol{I}_{k+1} - \Sigma_{i=1}^{p} \boldsymbol{\Phi}_i)$.[9]

Many analysts are only interested in the long run behavior of a single variable, y_t, in response to a set of exogenous regressors, which may or may not be endogenously related to each other. We have specified a system of equations such that each variable in the system responds to all others. In order to estimate a single equation for y_t we need to assume the $\boldsymbol{x}_t$ are weakly exogenous for the parameters of a conditional model for y_t that accounts for any contemporaneous correlations among y_t and $\boldsymbol{x}_t$. We thus partition $\boldsymbol{z}_t = (y_t, \boldsymbol{x_t'})'$ and the error, deterministic components, and coefficient matrices conformably. We then restrict the k-vector of coefficients on lagged levels of y_t in the equations for each $\boldsymbol{x}_t$ to be 0: $\boldsymbol{\pi}_{xy} = 0$ (PSS assumption 3). This eliminates the possibility of feedback from y_t to $\boldsymbol{x}_t$ and guarantees that any long run equilibrium involving y_t is unique. The marginal model for $\boldsymbol{x}_t$ is thus given by

$$\Delta \boldsymbol{x}_t = \boldsymbol{a}_{x0} + \boldsymbol{a}_{x1} t + \boldsymbol{\Pi}_{xx} \boldsymbol{x}_{t-1} + \Sigma_{i=1}^{p-1} \boldsymbol{\Gamma}_{xi} \Delta \boldsymbol{z}_{t-i} + \boldsymbol{\epsilon}_{xt}. \tag{3}$$

After conditioning on any contemporaneous correlation in the errors of y_t and $\boldsymbol{x}_t$, we can specify – and test hypotheses using – an ECM for y_t conditional on the $\boldsymbol{x}_t$:[10]

[7] Formally, the roots of $|\boldsymbol{\Phi}(L) = \boldsymbol{I}_{k+1} - \sum_{i=1}^{p} \boldsymbol{\Phi}_i z^j| = 0$ are either outside the unit circle $|z| = 1$ or on the unit circle $|z| = 1$.

[8] In order to allow the deterministic components of the model to contribute to the long run relationship, we must restrict μ and γ to be linear combination of the elements in the long run (cointegrating) vector. This implies that we must similarly restrict a_0 and a_1 in the VECM as follows:

$$\boldsymbol{a}_0 \equiv -\boldsymbol{\Gamma} \boldsymbol{\mu} + (\boldsymbol{\Gamma} + \boldsymbol{\Pi}) \boldsymbol{\gamma}, \boldsymbol{a}_1 \equiv -\boldsymbol{\Pi} \boldsymbol{\gamma}$$

[9] The short run matrix lag polynomial is given by $\boldsymbol{\Gamma}(L) \equiv \boldsymbol{I}_{k+1} - \Sigma_{i=1}^{p-1} \boldsymbol{\Gamma}_i L^i$, where $\boldsymbol{\Gamma}_i = -\Sigma_{j=i+1}^{p} \Phi_j$. The sum of the short-run coefficient matrixes in equation (2) $\boldsymbol{\Gamma} \equiv \boldsymbol{I}_m - \Sigma_{i=1}^{p-1} \boldsymbol{\Gamma}_i = -\boldsymbol{\Pi} + \Sigma_{i=1}^{p} i \boldsymbol{\Phi}_i$.

[10] In order for estimation of the conditional model to produce the same inferences as the full system given in equation (1) and equation (2), we must condition on any contemporaneous correlation between y_t and $\boldsymbol{x}_t$ contained

$$\Delta y_t = c_0 + c_1 t + \pi_{yy} y_{t-1} + \boldsymbol{\pi}_{yx.x}\boldsymbol{x}_{t-1} + \Sigma_{i=1}^{p-1}\boldsymbol{\psi}_i'\Delta\boldsymbol{z}_{t-i} + \boldsymbol{\omega}'\Delta\boldsymbol{x}_t + u_t{}^{11} \quad (5)$$

where $\boldsymbol{\pi}_{yx.x} \equiv \boldsymbol{\pi}_{yx} - \boldsymbol{\omega}'\boldsymbol{\Pi}_{xx}$. $\boldsymbol{\pi}_{yx}$ is a vector of coefficients describing the relationship between $\boldsymbol{x}_{t-1}$ and y_t. $\boldsymbol{\omega}$ describes the contemporaneous correlations among the variables in the system and $\boldsymbol{\Pi}_{xx}$ specifies the long run relationships among the $\boldsymbol{x}_t$. The quantity $\boldsymbol{\pi}_{yx} - \boldsymbol{\omega}'\boldsymbol{\Pi}_{xx}$ describes the effect of $\boldsymbol{x}_t$ on y_t. The equivalence of $\boldsymbol{\pi}_{yx.x}$ and $\boldsymbol{\pi}_{yx} - \boldsymbol{\omega}'\boldsymbol{\Pi}_{xx}$ highlights that the long run effects of $\boldsymbol{x}_t$ on y_t depend on the contemporaneous correlations imposed by the structure of $\boldsymbol{\omega}$.

Consistent with our uncertainty over the dynamics in $\boldsymbol{x}_t$, we wish to allow the $\boldsymbol{x}_t$ to be $I(0)$, $I(1)$ and not cointegrated, or $I(1)$ and cointegrated. This means the long run coefficient matrix for $\boldsymbol{x}_t$, $\boldsymbol{\Pi}_{xx}$, may have rank $0 \leq r_x \leq k$ (PSS Assumption 4). If $r_x = 0$, there are no cointegrating relationships and the $\boldsymbol{x}_t$ are purely $I(1)$. If $r_x = k$ (the number of independent variables in the system), the $\boldsymbol{x}_t$ are all $I(0)$. If $0 < r_x < k$, then there are r_x cointegrating relationships in $\boldsymbol{x}_t$.

Given our assumptions, if a long run relationship between y_t and $\boldsymbol{x}_t$ exists it is given by:

$$\pi_{yy} y_{t-1} + (\boldsymbol{\pi}_{yx} - \boldsymbol{\omega}'\boldsymbol{\Pi}_{xx})\boldsymbol{x}_{t-1}. \quad (6)$$

Conversely, there is no long run relationship between y_t and $\boldsymbol{x}_t$ *if and only if both* $\pi_{yy} = 0$ *and* $\boldsymbol{\pi}_{yx.x} = \boldsymbol{\pi}_{yx} - \boldsymbol{\phi}'\boldsymbol{\Pi}_{xx} = \mathbf{0}'$ for some k-vector $\boldsymbol{\phi}$ in which case the ECM reduces to a model in first differences.[12]

3 PSS Hypothesis Tests and their Limits

Given the model and assumptions described above, there is no long run relationship between y_t and $\boldsymbol{x}_t$ if $\pi_{yy} = 0$ **and** $\boldsymbol{\pi}_{yx.x} = \boldsymbol{\pi}_{yx} - \boldsymbol{\phi}'\boldsymbol{\Pi}_{xx} = \mathbf{0}'$. PSS are interested in the case where we know y_t is a unit root and there is no *cointegrating* relationship. Thus they propose to test the null hypothesis of no long run (cointegrating) relationship using a Wald or F-test where $H_{0F} : \pi_{yy} = \boldsymbol{\pi}_{yx.x} = 0$ against the alternative that *either or both* are nonzero: $H_{AF} : \pi_{yy} \neq 0$ or $\boldsymbol{\pi}_{yx.x} \neq 0$ or both.

in the error term $\boldsymbol{\epsilon}_t$. Following PSS, let the variance covariance matrix of the errors be given by $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_{yy} & \boldsymbol{\omega}_{yx} \\ \boldsymbol{\omega}_{xy} & \boldsymbol{\Omega}_{xx} \end{pmatrix}.$$

We can then express ϵ_{yt} in terms of the errors for the marginal model ($\boldsymbol{\epsilon}_{xt}$) as:

$$\epsilon_{yt} = \boldsymbol{\omega}_{yx}\boldsymbol{\Omega}_{xx}^{-1}\boldsymbol{\epsilon}_{xt} + u_t \quad (4)$$

where $u_t \sim IN(0, \omega_{uu})$ $\omega_{uu} \equiv \omega_{yy} - \boldsymbol{\omega}_{yx}\boldsymbol{\Omega}_{xx}^{-1}\boldsymbol{\omega}_{xy}$ and u is independent of $\boldsymbol{\epsilon}_{xt}$. Substitution of equation (4) into equation (2) produces the conditional model given in equation (5) above where $\boldsymbol{\omega} \equiv \boldsymbol{\Omega}_{xx}^{-1}\boldsymbol{\omega}_{xy}$ and $\boldsymbol{\psi}_i' \equiv \boldsymbol{\gamma}_{yi} - \boldsymbol{\omega}'\boldsymbol{\Gamma}_{xi}, i = 1, \ldots, p-1$..

[11]The constant and trend term are now modified to:

$$c_0 = -(\pi_{yy}, \boldsymbol{\pi}_{yx.x})\boldsymbol{\mu} + [\boldsymbol{\gamma}_{yx} + (\pi_{yy}, \boldsymbol{\pi}_{yx.x})]\boldsymbol{\gamma}, c_1 = -(\pi_{yy}, \boldsymbol{\pi}_{yx.x})\boldsymbol{\gamma}$$

[12]Given rank r_x, it follows that the rank of the long run coefficient matrix in the full system $rank\boldsymbol{\Pi}$ must be at least r_x and no more than $r_x + 1$. PSS further specify the conditions that must hold to ensure that the maximum order of integration in the system is one in each case. See Pesaran, Shin and Smith (2001) for further details.

Critical values for the test are unavailable for an arbitrary mix of $I(0)$ and $I(1)$ regressors. However, given the DGP and assumptions identified above, two polar cases – when $\boldsymbol{x}_t$ are all $I(0)$ and when $\boldsymbol{x}_t$ are all $I(1)$ – establish the bounds for the F-test. The lower bound is associated with $r_x = k$, in which case the $\boldsymbol{x}_t$ are all $I(0)$. The upper bound is associated with $r_x = 0$, in which case the $\boldsymbol{x}_t$ are all $I(1)$ and not cointegrated. Of course, the truth may lie between, in which case there is at least one cointegrating relationship among the $\boldsymbol{x}_t$.

To test for a long run cointegrating relationship the analyst estimates the conditional ECM, computes the F-statistic for the lagged levels variables, and compares the result to the bounds.[13] If F is below the lower bound, we cannot reject the null regardless of whether $\boldsymbol{x}_t \sim I(0)$, $I(1)$, or cointegrated. If F is greater than the upper bound then we can infer the existence of a long run relationship regardless of the dynamic properties of $\boldsymbol{x}_t$. If F is between the bounds, without knowing the dynamic properties of $\boldsymbol{x}_t$ we cannot determine whether to reject or fail to reject. If we knew the $\boldsymbol{x}_t$ were $I(0)$, then we would reject the null. If we knew $\boldsymbol{x}_t$ to be $I(1)$, then we would fail to reject.

Rejection of the null hypothesis does not gaurantee a *valid* long run equilibrium.[14] The alternative hypothesis is consistent with 4 types of long run relationships. PSS describe two of these relationships as *degenerate*: They are either nonsensical or the equilibrium is of a *simpler class* in which y_t has a long run equilibrium independent of $\boldsymbol{x}_t$. Degenerate equilibria occur when we reject the null hypothesis because either $\pi_{yy} \neq 0$ or $\boldsymbol{\pi}_{yx.x} \neq 0$ but not both; a non-degenerate relationship requires both $\pi_{yy} \neq 0$ and $\boldsymbol{\pi}_{yx.x} \neq 0$. We briefly describe each type of long run relationship permitted under the alternative hypothesis and present the possible relationships between y_t and $\boldsymbol{x}_t$ in Table 1.

[Table 1 Here]

The first two alternatives, A_1 and A_2, describe degenerate long run relationships. Under Alternative A_1, $\pi_{yy} = 0$ and $\boldsymbol{\pi}_{yx.x} \neq 0$ and the long run relationship given in equation (6) is reduced to:

$$(\boldsymbol{\pi}_{yx} - \boldsymbol{\omega}'\boldsymbol{\Pi}_{xx})\boldsymbol{x}_{t-1}. \tag{7}$$

In this case y_t is a unit root process but is not cointegrated with $\boldsymbol{x}_t$. Instead, Δy_t is influenced in the short run by the stationary linear combination of cointegrated $\boldsymbol{x}_t$ or the $\boldsymbol{x}_t$ are all stationary. [15]

If alternative A_2 holds, $\pi_{yy} \neq 0$ and $\boldsymbol{\pi}_{yx.x} = 0$. In this case, y_t is stationary (or trend stationary) and independent of $\boldsymbol{x}_t$ in the long run, regardless of the dynamic properties of $\boldsymbol{x}_t$ such that the long run behavior of y_t reduces to:

$$\pi_{yy} y_{t-1}. \tag{8}$$

[13]Different sets of critical values apply given the deterministic relationship specified and allow for constants and trends to be omitted, unrestricted, or to apply to the long run relationship. More specifically, if there is no constant and trend in the original VAR such that $\mu = \gamma = 0$, then $c_0 = c_1 = 0$ (PSS Case 1). If $\mu \neq 0$ but $\gamma = 0$, then we may impose restrictions on c_0 such that the constant is part of the long run relationship ($c_0 = -(\pi_{yy}, \boldsymbol{\pi}_{yx.x})\boldsymbol{\mu}$, PSS case 2) or is unrestricted ($c_0 \neq 0$, PSS case 3). In both of these cases γ and thus c_1 equal zero. If the trend (γ) is restricted to be part of the long run relationship then c_1 is restricted to be $c_1 = -(\pi_{yy}, \boldsymbol{\pi}_{yx.x})\boldsymbol{\gamma}$ while the constant is unrestricted (PSS case 4). Finally, if we impose no restrictions on the constant or trend, $\mu \neq 0$ and $\gamma \neq 0$ such that $c_0 \neq 0$ and $c_1 \neq 0$ (PSS case 5).

[14]PSS acknowledge this possibility on page 295.

[15]Note, if either $\boldsymbol{\omega} = 0$ (there are no contemporaneous correlations between y_t and $\boldsymbol{x}_t$) or $\boldsymbol{x}_t$ are unit roots but not cointegrated ($r_x = 0$ such that $\boldsymbol{\Pi}_{xx} = 0$), the long run relationship is given by the coefficients that describe the relationships among the $\boldsymbol{x}_{t-1}$ and y_t, $\boldsymbol{\pi}_{yx}$.

Changes in $\boldsymbol{x}_t$ may affect changes in y_t in the short run but y_t returns to its unconditional mean in the long run.

The remaining specifications characterize non-degenerate long run equilibria between y_t and $\boldsymbol{x}_t$ that typically motivate our hypothesis tests. In particular, both $\pi_{yy} \neq 0$ and $\boldsymbol{\pi}_{yx.x} = \boldsymbol{\pi}_{yx} - \boldsymbol{\phi}'\boldsymbol{\Pi}_{xx} \neq \mathbf{0}'$ and the long run relationship is given by equation (6). In the first of these (Alternative A_{3a}), y_t is a unit root process and cointegrated with $\boldsymbol{x}_t$, which may also be cointegrated or $I(1)$ processes. A second type of non-degenerate equilibrium holds when y_t is stationary and dependent on $\boldsymbol{x}_t$ (Alternative A_{3b}). In this case the $\boldsymbol{x}_t$ may be stationary or cointegrated, but in either case, their influence on y_t is via a linear combination of the $\boldsymbol{x}_t$ that is stationary.[16]

The F-test leaves open the possibility that y_t is or is not a function of $\boldsymbol{x}_t$ in the long run and that any long run equilibrium is or is not degenerate. As such, PSS propose using the familiar ECM test for cointegration, $H_{0t} : \pi_{yy} = 0$, as a way to arbitrate among a subset of the alternatives discussed above and specifically to determine whether $H_{A_1,F}$ characterizes the data when $H_{0F} = 0$ is rejected.[17] Like the F-test, critical values for this t-test depend on the nature of $\boldsymbol{x}_t$. PSS derive lower and upper bounds consistent with $\boldsymbol{X} \sim I(0)$ and $\boldsymbol{X} \sim I(1)$, respectively. If we fail to reject the null, then rejection of the F-test implies $H_{A_1,F}$ holds and the long run equilibrium is undefined. If we reject both null hypotheses, we know only that either $H_{A_2,F}$, $H_{A_{3a},F}$ or $H_{A_{3b},F}$ holds. We can only rule out A_2 and A_{3b} when we are certain y_t is $I(1)$ (as PSS assume). This presents a dilemma for the analyst who is uncertain of the dynamics of y_t.

Political scientists frequently use the ECM to test for the existence of long run relationships, primarily relying on the test of the null $H_0 : \pi_{yy} = 0$ and using either MacKinnon critical values (CITE) or the PSS critical values for inference (CITE). In many of these cases, y_t may be stationary, but inferences are often drawn without appreciation of the different types of long run behavior that may lead to rejection of the null and without appreciation of the complications stationary y_t present for valid inference. If y_t is stationary, we will (correctly) reject the null $H_{0F} : \pi_{yy} = \boldsymbol{\pi}_{yx.x} = 0$ and the null $H_{0t} : \pi_{yy} = 0$ and do so with increasing confidence as the persistence in y_t decreases. This is true for MacKinnon critical values and PSS critical values. Rejection of the null hypothesis given a stationary y_t does not necessarily imply a non-degenerate long run relationship between y_t and $\boldsymbol{x}_t$. The long run behavior of y_t may or may not depend on $\boldsymbol{x}_t$. Alternative A_2 or A_{3b} may apply.

The above discussion makes clear that if both theory and univariate tests are inconclusive as to whether y_t is $I(0)$ or $I(1)$, it is a dangerous strategy to conclude a long run relationship exists between y_t and $\boldsymbol{x}_t$ based on either the ECM test or the F-test proposed by PSS. We need to be able to rule out both A_1 and A_2 to be certain the equilibrium relationship is non-degenerate. We cannot do so if the dynamics of y_t are uncertain.

[16] The distinction between cases 3a and 3b depends on the rank of the rank factorized matrix of long run relationships. Specifically, we can write $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$, where $\boldsymbol{\alpha}$ give the speed of adjustment to disequilibrium – the error correction coefficient – and $\boldsymbol{\beta}$ comprise the the vector of long run relationships. If $rank(\boldsymbol{\beta}_{yx}, \boldsymbol{\beta}_{xx}) = r_x$, the vector involving y_t duplicates the long run relationships in $\boldsymbol{x}_t$ and all series are stationary. If $rank(\boldsymbol{\beta}_{yx}, \boldsymbol{\beta}_{xx}) = r_x + 1$ then the vector contains independent information and all series are unit roots and y_t is cointegrated with $\boldsymbol{x}_t$. See *Eviews Econometric Analysis Insight Blog: AutoRegressive Distributed Lag (ARDL) Estimation. Part 2 - Inference* (2017) for further details.

[17] Banerjee, Dolado and Mestre (1998) first developed the test but these authors only considered the case where $\boldsymbol{x}_t$ were all unit roots ($rank\boldsymbol{\Pi}_{xx} = 0$).

4 The LRM Test

What should analysts do if they are uncertain whether y_t is a stationary or unit root process? For many applications in political science the analyst is a) uncertain whether their data (including y_t) are truly $I(0)$ or $I(1)$ and b) less interested in inference on the null hypothesis of no cointegration than in inference regarding the existence of a valid or non-degenerate long run relationship. Thus, we would like to be able to test the null that at least one of π_{yy} *and* $\boldsymbol{\pi}_{yx.x} = 0$ against the alternative that both are nonzero: $\pi_{yy} \neq$ *and* $\boldsymbol{\pi}_{yx.x} \neq 0$.

We propose a test for the existence of a non-degenerate long run relationship between y_t and $\boldsymbol{x}_t$ based on the long run multiplier (LRM). We can express the conditional ECM in equation (5) such that the long run relationship is isolated (in parentheses) as follows:

$$\Delta y_t = c_0 + c_1 t + \pi_{yy}(y_{t-1} + \frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}}\boldsymbol{x}_{t-1}) + \Sigma_{i=1}^{p-1}\boldsymbol{\psi}_i'\Delta\boldsymbol{z}_{t-i} + \boldsymbol{\omega}'\Delta\boldsymbol{x}_t + \mu_t. \tag{9}$$

where $(y_{t-1} + \frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}}\boldsymbol{x}_{t-1})$ gives the long run, and possibly cointegrating, relationship, π_{yy} gives the rate of return to equilibrium, and $\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}}$ is the long run multiplier.

In order for a non-degenerate long run relationship to exist, *both* π_{yy} *and* $\boldsymbol{\pi}_{yx.x}$ must be nonzero. This implies that $\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}}$ must also be non-zero. If this is true, $\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}}$ describes the link between $\boldsymbol{x}$ and y_t and π_{yy} tells us how this linkage drives change in y_t. In contrast, if $\pi_{yy} = 0$ the entire equilibrium term drops out of the equation and $\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}}$ is undefined. If $\boldsymbol{\pi}_{yx.x} = 0$, the LRM is zero and y_t is not a function of $\boldsymbol{x}_t$. *Thus, a non-degenerate or valid equilibrium relationship between* y_t *and* $\boldsymbol{x}_t$ *requires the LRM to be nonzero.*

This test applies whether y_t is $I(0)$ or $I(1)$. If y_t is a unit root process, the only way π_{yy} can be nonzero is if y_t is linked to $\boldsymbol{x}_t$ in the long run. In other words, it must be the case that $\boldsymbol{\pi}_{yx.x} \neq 0$ and $\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}} \neq 0$ such that y_t has a long run, cointegrating relationship with $\boldsymbol{x}_t$. This is the logic underlying the ECM test for cointegration (Banerjee, Dolado and Mestre 1998; Banerjee et al. 1993; Ericsson and MacKinnon 2002). If y_t is a unit root and $\pi_{yy} \neq 0$ it must be cointegrated with $\boldsymbol{x}_t$.

In the stationary case, π_{yy} is, by definition, nonzero. y_t will always return to its mean in the long run, whether that mean is conditional on $\boldsymbol{x}_t$ or not. (If $\pi_{yy} = 0$, y_t is a unit root.) Only if $\boldsymbol{\pi}_{yx.x}$, and thus $\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}} \neq 0$, will the long run value of y_t be conditional on $\boldsymbol{x}_t$.[18]

The central insight is that a) in order for the LRM to exist, π_{yy} must be nonzero, $\pi_{yy} = 0$ renders the LRM undefined; and b) if the LRM is nonzero it must be the case that $\boldsymbol{\pi}_{yx.x}$ is nonzero. Thus, inference about the existence of a long run relationship between y_t and $\boldsymbol{x}_t$ – in either the stationary or unit root case – can be made based on the hypothesis test $H_0 : LRM = 0$.[19] Put differently, we cannot reject the null $H_{0,LRM} : \frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}} = 0$ if there is no long run relationship between y_t and $\boldsymbol{x}_t$ (π_{yy} and $\boldsymbol{\pi}_{yx.x} = 0$) or if there is a degenerate long run relationship between y_t and $\boldsymbol{x}_t$ ($\pi_{yy} = 0$ and $\boldsymbol{\pi}_{yx.x} \neq 0$ or $\pi_{yy} \neq 0$ and $\boldsymbol{\pi}_{yx.x} = 0$). If $\pi_{yy} = 0$ the LRM will be undefined, regardless of $\boldsymbol{\pi}_{yx.x}$, and if $\boldsymbol{\pi}_{yx.x} = 0$, the LRM will be zero.

We present the null and alternative hypotheses in Table 2. If we cannot reject the null, then y_t does not have a valid (non-degenerate) equilibrium with $\boldsymbol{x}_t$. This holds whether the data are unit root or stationary processes and whether the regressors are cointegrated. If we can reject the null, we infer a long run relationship between y_t and $\boldsymbol{x}_t$. We cannot distinguish

[18] This is, of course, a common occurrence. But as we will see below, the appropriate critical values on the hypothesis test on the LRM will often be nonstandard, a fact that has eluded most applied research.

[19] It is also possible that the LRM is undefined.

a long run cointegrating relationship, $H_{A_1,LRM}$, from a long run relationship between a set of stationary variables, $H_{A_2,LRM}$, using this test, but this is usually not of concern.

[Table 2 Here]

The LRM is not estimated directly in the ECM (or the equivalent ADL). While we can calculate the LRM from the ECM (or, identically, in terms of the parameters of an ADL), we cannot calculate the standard error for the LRM. There is no simple formula for calculating the standard error because the LRM is expressed as a ratio of coefficients. Various methods exist for approximating the variance of a quotient of items with known variances. The simplest practice is to estimate the Bewley transformation of the model using instrumental variables regression 1979. The LRM is directly estimated along with its standard error.[20,21] The Bewley transformation for the general case with a constant and trend is written as:

$$y_t = \phi_0 + \tau t - \phi_1 \Delta y_t + \psi_0 \boldsymbol{x}_t - \psi_1 \Delta \boldsymbol{x}_t + \mu_t \tag{10}$$

where $\psi_0 = -\frac{\pi_{yx}}{\pi_{yy}}$, the LRM, $\phi_0 = -\frac{c_0}{\pi_{yy}}$, $\tau = \frac{c_1}{\pi_{yy}}$, $\phi_1 = -\frac{\pi_{yy}+1}{\pi_{yy}}$, $\psi_1 = \pi_{yx}$, and $\mu = -\frac{e}{\pi_{yy}}$ in the conditional ECM. A constant, trend, x_t, x_{t-1}, and y_{t-1} are used as instruments to estimate the model (Banerjee et al. 1993; De Boef and Keele 2008).

It is not obvious what form the distribution of the t-test on the LRM will have. If y_t is a unit root the LRM test statistic is likely to have a nonstandard distribution, much like the t and F-tests evaluated by PSS. It is also not clear how the sample size, presence or absence of deterministic components, or number of regressors will affect the distribution.[22] In the next section we calculate the appropriate critical values for the LRM test as each of these features of the data and model vary and we offer a bounds testing framework for inference on the LRM.

Before doing so, we note three advantages to focusing on the significance of the LRM as a test for long run equilibrium. First, inferences about the existence of a long run equilibrium relationship between y_t and $\boldsymbol{x}_t$ do not depend on the dynamic properties of the individual time series. Second, the LRM test has a specific advantage over the ECM test for cointegration ($H_0 : \pi_{yy} = 0$) when we know y_t is $I(1)$ and we have multiple independent variables in the model. While rejecting the null $H_0 : \pi_{yy} = 0$ implies y_t is cointegrated with the vector of $\boldsymbol{x}_t$, it does not tell us which $\boldsymbol{x}_t$ contribute to the cointegrating relationship. In contrast, the LRM test allows us to draw inferences about whether there is a long run relationship between y_t and any given $\boldsymbol{x}_t$. Third, and perhaps most importantly, rejection of the null implies a non-degenerate long run relationship exists between y_t and $\boldsymbol{x}_t$.

5 Inference Using the LRM Test

PSS derive the asymptotic distribution of critical values for both of their recommended hypothesis tests under the assumption that all $\boldsymbol{x}_t$ are $I(0)$ or all are $I(1)$. Their simulated test statistics provide lower and upper bounds for the appropriate critical values. They also show that these bounds depend on the number of independent variables k, the sample size T, and the specification of the deterministic components included in the model. They provide approximate asymptotic critical values for the upper and lower bounds for both the t- and F-tests for

[20] Banerjee et al 1993, drawing on Wickens and Breusch 1988, prove that estimates of the standard error based on approximations applied to either the ADL and ECM are equivalent to the estimate obtained directly from estimating the Bewley transformation using instrumental variables.

[21] One could also generate estimates of the standard error of the LRM using the Delta method or bootstrapping.

[22] (Pesaran and Shin 1998) show that the distribution of the estimate of the LRM itself is mixture normal asymptotically and that it is super consistent.

$k = 0, 1, 2, \ldots 10$ and 5 possible specifications of the deterministic components using stochastic simulations with $T = 1000$ and 40,000 Monte Carlo replications.[23]

We compute the asymptotic distribution of the LRM test using the same procedure. We estimate the sampling distribution of the LRM test statistic in the Bewley IV regression given in equation (10) under the true null hypothesis that there is no long run relationship between y_t and $\boldsymbol{x}_t$ ($\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}} = 0$). Following PSS, we simulate the distribution under a range of conditions. Given these sampling distributions we calculate the value of the LRM test statistic associated with the 95th percentile of the distribution. This value represents the critical value for the test statistic under the simulated conditions.

First, we consider the simple case in which $k = 1$. Our analysis illustrates the need for a bounds hypothesis testing framework on the LRM t-test. Then, we extend our analysis to $k > 1$, allowing the dynamic properties of $\boldsymbol{x}_t$ to vary and considering different treatment of the deterministic features of the model.

5.1 Demonstrating the Need for Nonstandard Critical Values: The Importance of Autocorrelation and the Existence of Bounds

Implicit in the adoption of a bounds testing framework is the acceptance of the fact that a) we are uncertain about the underlying dynamic properties of the data and b) that appropriate critical values depend on these properties. Thus we begin by demonstrating that the appropriate critical values for the LRM t-test depend on the degree of autocorrelation in the data and are bounded by the cases where y_t is $I(1)$ and x_t is $I(1)$ and where y_t is $I(1)$ and x_t is $I(0)$.

We turn to stochastic simulations to generate critical values for the LRM t-test for varying degrees of autocorrelation in y_t and x_t for $k = 1$ and sample sizes $T = 75$ and $T = 1000$. The smaller sample size is common in applied work while the larger sample size produces critical values that approximate the asymptotic distribution. Specifically, we generate 2 independent autoregressive processes, $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_t = \rho_x x_{t-1} + e_{xt}$ with the errors drawn from independent standard normal distributions. We vary the values of ρ_y and ρ_x from 0 to 9.0 in increments of 0.10 and from 0.90 to 1.0 in increments of 0.01. For each combination of ρ_y and ρ_x, we simulate the sampling distribution of the LRM t-statistic using 40,000 replications. The LRM t-value is estimated as the t-value on x_t in the Bewley ECM given in equation (10) above in a model with an unrestricted constant and no trend. Figure 1 presents the simulated critical values associated with the 95th percentile of the distribution for values of ρ_x as ρ_y varies for both sample sizes.

[Figure 1 Here]

We draw three conclusions from our results. First, looking within each panel of the figure, we see that for white noise y_t ($\rho_y = 0$), standard critical values apply regardless of the dynamics in x_t: all lines cross the y-axis at approximately 1.96. However, as y_t becomes more autoregressive, the appropriate critical values fan out from the standard normal critical values and depend on the degree of autocorrelation in x_t (ρ_x). For smaller values of ρ_x, the estimated critical values are closer to zero than the standard normal critical values while for larger values

[23] They consider a) no constant and no trend, b) a restricted constant and no trend, c) an unrestricted constant and no trend, d) a restricted constant and trend, and e) an unrestricted constant and unrestricted trend. These cases allow for the deterministic components, when included in the model, to either be included in the long run relationship, in which case they are restricted to be a linear combination of any cointegrating vector, or to be separate from the long run relationship (does that make sense if y_t is not $I(1)$?).

of ρ_x they are further from zero. For example, when $T = 75$, $\rho_y = 0.70$, and $\rho_x = 0$, the simulated critical value is approximately 1.85 while when $\rho_x = 0.80$, it increases to 2.25 and when x_t is a unit root the estimated t-value must be greater than 2.7 to reject the null.

Second, looking across the panels, we see that while the pattern is the same for each sample size, the range of autocorrelation in either series for which the critical values depart from the standard normal is considerably smaller in the larger sample. Still, even with sample sizes that exceed most published work in political science, if the DGP for x_t and y_t are strongly autoregressive, the danger of incorrect inference for any given p-value from standard normal critical values remains. With smaller sample sizes, the danger of incorrect inference extends to even moderate levels of autocorrelation in either series.

Finally, the results establish a lower and upper bound for the t-statistic that applies regardless of sample size. A lower bound of approximately 1.4 is established when y_t is a unit root ($\rho_y = 1.0$) and x_t is white noise ($\rho_x = 0$) while an upper bound of approximately 3.7 is established when both y_t and x_t are unit roots ($\rho_y = \rho_x = 1.0$). Thus, if the LRM t-statistic is greater than 3.7, we can infer a long run relationship exists between y_t and x_t while if it is smaller than 1.4, we can infer there is not a long run relationship between y_t and x_t. For t-statistics in between these bounds, we need to know the sample size and the degree of autocorrelation in both time series to draw an inference.

These results affirm that the range of autocorrelation in y_t for which standard critical values are appropriate can be quite small when sample sizes take on values typical of much applied work and suggest the possibility that our confidence in some published findings regarding the significance of the LRM may be overstated. In the next section we generalize our findings and establish bounds for the LRM t-statistic for $k = 1, 2, 3, 4$ as the dynamic behavior of $\boldsymbol{x}_t$ and the specification of deterministic components vary.

5.2 Generalizing the Bounds

Critical values could be calculated for the LRM t-statistic if the dynamic properties of the data were known, but this information is not available in applied settings. In this section we show how the bounds for the LRM t-statistic are set and how the bounds change for different k and different deterministic components.

Table 3 shows quantiles for the empirical distributions of the ECM t-tests. The test statistics were produced from a model with three independent variables ($k = 3$). The rows of the table show the different possible permutations of I(0) and I(1) variables. These permutations range from the case where all the variables are white noise I(0) to the case where all the variables are unit roots I(1). The 2.5 and 97.5 quantiles are shown for each t-test for in each model. We simulated the sampling distributions of the t-statistics using 10,000 replications of $T = 1,000$.

[Table 3 Here]

The first column shows the quantiles for π_{yy}. These are the same quantiles reported by PSS. The t-statistics in the top half of the table are noticeably larger in the top half of the table. This is because $\rho_y = 0$ in these models. The empirical t-statistics are large because $\pi \neq 0$. The t-statistics for π_{yy} in the lower part of the table are more interesting. These are (roughly) the critical values presented by PSS. The lower bound for the t-statistic is set by the case where all the independent variables are I(0) and y_t is I(1). The upper bound for the

t-statistic is set by the case where all the independent variables are I(1) and the y_t is I(1). Our bounds (-3.12 and -4.01) are near the bounds reported by PSS (-3.13 and -4.05).[24]

Several results stand out for LRM t-statistics. First, the empirical distributions for the LRM t-statistics are (roughly) symmetric where the the empirical distributions for the π_{yy} t-statistics are not. Second, the critical values for the cases where $\rho_y = 0$ have standard distributions. This is consistent with the results presented in 1. Third, the critical values are non-standard for cases where $\rho_y = 1$. This is also consistent with the results presented in Figure 1. The results presented in Table 3 also show that the shapes of these non-standard distributions change with the number of I(1) variables in the model. This further complicates the task of computing specific critical values. The potential for error increases with the number of independent variables.

The bounds for the LRM t-statistics follow a different pattern than the bounds for the π_{yy} t-statistics. Where the bounds for π_{yy} are set by the cases where all the independent variables are either I(0) or I(1), the bounds for the LRMs are set by the case where all the independent variables are I(0) and y_t is I(1) and the case where one of the independent variables is I(1) and y_t is I(1). The lowest LRM t-statistics (1.30) are in the row where all the independent variables are I(0). This is consistent across variables. The highest values change from column to column based on the location of the single I(1) variable. Consider the case where $\rho_{x1} = \rho_{x2} = 0$ and $\rho_{x3} = \rho_y = 1$. The t-statistics for ρ_{x1} and ρ_{x2} are (roughly) equal at 1.38. The t-statistic for ρ_{x3} is much higher (3.70). This same pattern exists in each of the columns. The I(1) independent variables in the models where there are two I(0) stationary variables have the largest t-statistics. All the other t-statistics fall between this upper bound and the lower bound set by the case where all the independent variables are I(0) and y_t is I(1). This includes all the standard t-statistics in the top half of the table for the models where $\rho_y = 0$. A final result that bares mentioning is the similarity of the t-statistics across the independent variables. The practical consequence of this similarity is that the bounds can be derived from one of the independent variables.

Why are the models that set the bounds different for π_{yy} and the LRMs? The lower bounds are set by the same model. The upper bounds are different because of spurious correlation among the multiple independent variables. Even though the series are generated as independent processes, spurious correlation among the independent variables occurs when there are multiple I(1) variables in the model. This correlation affects the estimates. The decreasing maxima among the I(1) t-statistics as the number of I(1) variables increases is evidence of this phenomenon.

Table 4 shows how the bounds for the LRMs change as T and k increase. The first three columns show the upper (UB) and lower (UB) bounds for the t-statistics for $T = 75, 150,$ and $1,000$. The fourth and fifth columns show the means and variances of these distributions ($T = 1,000$). The rows show the changes in the bounds as the number of independent variables increases $k = 1, 2, 3, 4$. We simulated the sampling distributions of the t-statistics using 10,000 replications of each sample size.

[Table 4 Here]

The bounds don't change as T increases. The bounds don't change as k increases.[25]. This suggests the bounds presented in Table 3 can be used regardless of the sample size or the number of independent variables.[26]

[24]See Table CII(iii) Case III page 303.

[25]This result does not change when we increase the number of simulations to 40,000

[26]We need to check this result with smaller sample sizes.

Table 5 (not in this draft) shows how the bounds change with different deterministics.

6 Application

7 Conclusion

Time series statistics texts teach us to begin analysis by conducting pre-tests to determine the dynamic properties of our time series. Such tests, one is led to believe, produce clear cut inferences about which we are highly confident and that neatly proscribe the appropriate modeling and hypothesis testing strategy. Practitioners of time series analysis, however, appreciate the difficulty of the task and the tendency for inferences to be uncertain in the first stage and the path forward unclear. One doesn't need to conduct many time series analyses to know that the textbook cases are not the typical case encountered in practice. In particular, time series are seldom unambiguously $I(0)$ or $I(1)$ and often times they are neither all (again unambiguously) $I(0)$ nor all $I(1)$.

Economists have focused attention on the development of methods for unit root time series. Given the nature of their data and theory, this makes sense; they are frequently confident their time series contain unit roots. Political time series behave differently than most economic time series and our theories seldom speak definitively about persistence. Unit roots, it seems, are much more rare in political science. But often pre-testing – and even theory – suggest our time series are highly persistent. As we argue above, highly persistent, but not unit root, time series present problems for the standard textbook approaches to testing for long run relationships. We offer the following suggestions. First, admit the uncertainty hypotheses conducted behind closed doors suggest. Second, adopt a modeling strategy and hypothesis tests that account for it. A bounds testing framework like that first proposed by PSS and adopted here move in that direction. But in addition, the analyst needs to be fully aware of the assumptions that underlie hypothesis tests and the meaning of the alternative hypothesis under different assumptions.

We have argued that uncertainty about univariate dynamics needs to be recognized, even embraced, in the modeling strategies we adopt. This recognition has led us to propose the LRM t-test and a bounds testing framework. We have only begun to investigate the distributional behavior of the test statistic. Still to come is analysis of the effects of different specifications of the deterministic components and the consequences of cointegration in $\boldsymbol{x}_t$ on the behavior of the test statistic.

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Table 1: The PSS F-test

	Specification	Conclusion	Integration Order (y_t)
$H_{0,F}$	$\pi_{yy} = 0$ and $\boldsymbol{\pi}_{yx.x} = 0$	No equilibrating relationship	$y_t \sim I(1)$
$H_{A_1,F}$	$\pi_{yy} = 0$ and $\boldsymbol{\pi}_{yx.x} \neq 0$	Nonsense equilibrating relationship	$y_t \sim I(1)$
$H_{A_2,F}$	$\pi_{yy} \neq 0$ and $\boldsymbol{\pi}_{yx.x} = 0$	Degenerate equilibrating relationship	$y_t \sim I(0)$ or trend stationary
$H_{A_{3a},F}$	$\pi_{yy} \neq 0$ and $\boldsymbol{\pi}_{yx.x} \neq 0$	Nondegenerate cointegrating equilibrating relationship	$y_t \sim I(1)$
$H_{A_{3b},F}$	$\pi_{yy} \neq 0$ and $\boldsymbol{\pi}_{yx.x} \neq 0$	Nondegenerate equilibrating relationship	$y_t \sim I(0)$

Note: $H_{0,F}$ is based on the conditional ECM given in equation (5). This table was adapted from *Eviews Econometric Analysis Insight Blog: AutoRegressive Distributed Lag (ARDL) Estimation. Part 2 - Inference* (2017).

Table 2: The LRM test

	Specification	Conclusion	Integration Order (y_t)
$H_{0,LRM}$	$\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}} = 0$	No equilibrating relationship	$y_t \sim I(1)$ or $I(0)$
$H_{A_1,LRM}$	$\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}} \neq 0$	Nondegenerate cointegrating equilibrating relationship	$y_t \sim I(1)$
$H_{A_2,LRM}$	$\frac{\boldsymbol{\pi}_{yx.x}}{\pi_{yy}} \neq 0$	Nondegenerate equilibrating relationship	$y_t \sim I(0)$

Note: $H_{0,LRM}$ is based on the Bewley instrumental variables regression given in equation (10).

Table 3: The Empirical Distribution of the ECM t-test and Simulated Critical Values for the LRM t-test: Identifying the Bounds Conditions

	y_{t-1}		$x_{1,t-1}$		$x_{2,t-1}$		$x_{3,t-1}$	
	2.5%	97.5%	2.5%	97.5%	2.5%	97.5%	2.5%	97.5%
$\rho_{x1}=0 \mid \rho_{x2}=0 \mid \rho_{x3}=0 \mid \rho_y=0$	-33.57	-29.62	-1.96	1.93	-1.91	1.96	-1.90	1.97
$\rho_{x1}=1 \mid \rho_{x2}=0 \mid \rho_{x3}=0 \mid \rho_y=0$	-33.61	-29.64	-1.97	1.93	-1.92	1.95	-1.91	1.98
$\rho_{x1}=0 \mid \rho_{x2}=1 \mid \rho_{x3}=0 \mid \rho_y=0$	-33.61	-29.64	-1.98	1.94	-1.94	1.98	-1.90	1.98
$\rho_{x1}=0 \mid \rho_{x2}=0 \mid \rho_{x3}=1 \mid \rho_y=0$	-33.62	-29.65	-1.94	1.92	-1.91	1.96	-1.94	1.96
$\rho_{x1}=1 \mid \rho_{x2}=1 \mid \rho_{x3}=0 \mid \rho_y=0$	-33.63	-29.67	-2.00	1.99	-1.94	1.95	-1.92	1.98
$\rho_{x1}=0 \mid \rho_{x2}=1 \mid \rho_{x3}=1 \mid \rho_y=0$	-33.64	-29.67	-1.97	1.93	-1.94	1.94	-1.96	1.98
$\rho_{x1}=1 \mid \rho_{x2}=0 \mid \rho_{x3}=1 \mid \rho_y=0$	-33.64	-29.68	-2.03	1.97	-1.93	1.95	-1.97	1.99
$\rho_{x1}=1 \mid \rho_{x2}=1 \mid \rho_{x3}=1 \mid \rho_y=0$	-33.67	-29.71	-2.04	2.01	-1.94	1.95	-1.97	1.99
$\rho_{x1}=0 \mid \rho_{x2}=0 \mid \rho_{x3}=0 \mid \rho_y=1$	-3.12	0.23	-1.30	1.29	-1.29	1.30	-1.27	1.31
$\rho_{x1}=1 \mid \rho_{x2}=0 \mid \rho_{x3}=0 \mid \rho_y=1$	-3.47	0.06	-3.60	3.69	-1.36	1.35	-1.34	1.40
$\rho_{x1}=0 \mid \rho_{x2}=1 \mid \rho_{x3}=0 \mid \rho_y=1$	-3.44	0.07	-1.37	1.38	-3.63	3.60	-1.35	1.40
$\rho_{x1}=0 \mid \rho_{x2}=0 \mid \rho_{x3}=1 \mid \rho_y=1$	-3.46	0.09	-1.38	1.36	-1.35	1.38	-3.56	3.70
$\rho_{x1}=1 \mid \rho_{x2}=1 \mid \rho_{x3}=0 \mid \rho_y=1$	-3.73	-0.11	-3.41	3.43	-3.38	3.36	-1.40	1.45
$\rho_{x1}=0 \mid \rho_{x2}=1 \mid \rho_{x3}=1 \mid \rho_y=1$	-3.73	-0.12	-1.44	1.44	-3.51	3.40	-3.46	3.41
$\rho_{x1}=1 \mid \rho_{x2}=0 \mid \rho_{x3}=1 \mid \rho_y=1$	-3.77	-0.12	-3.48	3.52	-1.42	1.43	-3.45	3.50
$\rho_{x1}=1 \mid \rho_{x2}=1 \mid \rho_{x3}=1 \mid \rho_y=1$	-4.01	-0.32	-3.24	3.32	-3.25	3.31	-3.29	3.32

Note: Critical values are computed via stochastic simulations using 10,000 replications for the LRM t-statistic in the Bewley instrumental variables regression:

$$y_t = \phi_0 - \phi_1 \Delta y_t + \psi_0 \boldsymbol{x}_t - \psi_1 \Delta \boldsymbol{x}_t + \mu_t$$

where $\psi_0 = -\frac{\pi_{yx}}{\pi_{yy}}$, the LRM, $\phi_0 = -\frac{c_0}{\pi_{yy}}$, $\phi_1 = -\frac{\pi_{yy}+1}{\pi_{yy}}$, $\psi_1 = \pi_{yx}$, and $\mu = -\frac{e}{\pi_{yy}}$ in the conditional ECM. A constant x_t, x_{t-1}, and y_{t-1} are used as instruments to estimate the model (Banerjee et al. 1993; De Boef and Keele 2008). The time series y_t and x_t are generated from: $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_t = \rho_x x_{t-1} + e_{xt}$ where the errors are drawn from independent standard normal distributions.

Table 4: Upper and Lower Bounds for the LRM t-test by k and T

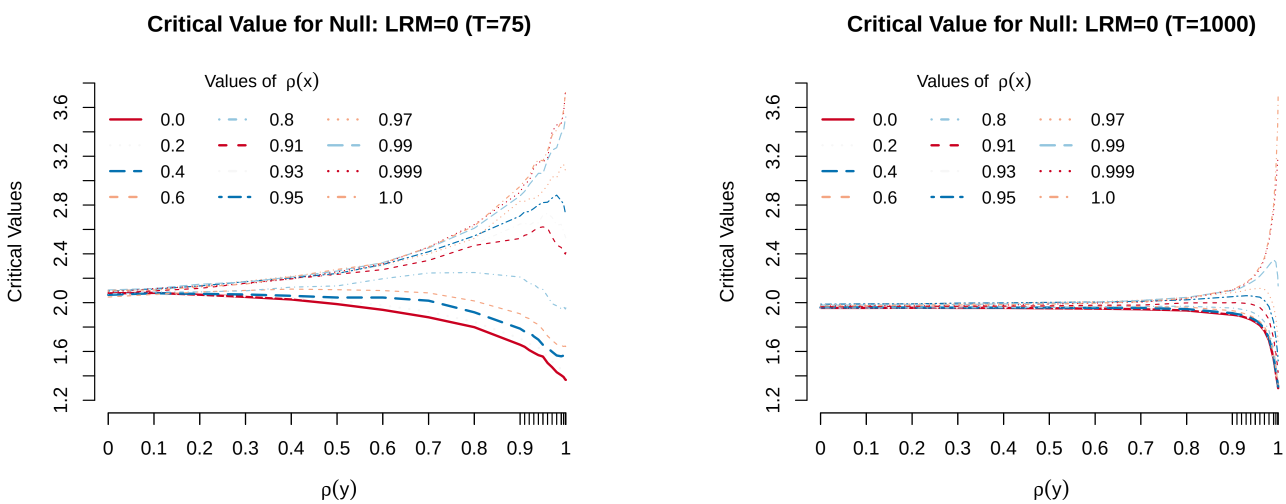
	$T = 75$		$T = 150$		$T = 1000$		Mean		Variance	
k	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
1	1.39	3.71	1.35	3.73	1.31	3.70	X	Y		
2	1.39	3.56	1.32	3.68	1.30	3.65	X	Y		
3	1.40	3.58	1.35	3.51	1.30	3.69	X	Y		
4	1.39	3.60	1.34	3.63	1.30	3.62	X	Y		

Note: Critical values are computed via stochastic simulations using 10,000 replications for the LRM t-statistic in the Bewley instrumental variables regression:

$$y_t = \phi_0 - \phi_1 \Delta y_t + \psi_0 \boldsymbol{x}_t - \psi_1 \Delta \boldsymbol{x}_t + \mu_t$$

where $\psi_0 = -\frac{\pi_{yx}}{\pi_{yy}}$, the LRM, $\phi_0 = -\frac{c_0}{\pi_{yy}}$, $\phi_1 = -\frac{\pi_{yy}+1}{\pi_{yy}}$, $\psi_1 = \pi_{yx}$, and $\mu = -\frac{e}{\pi_{yy}}$ in the conditional ECM. A constant x_t, x_{t-1}, and y_{t-1} are used as instruments to estimate the model (Banerjee et al. 1993; De Boef and Keele 2008). The time series y_t and x_t are generated from: $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_t = \rho_x x_{t-1} + e_{xt}$ where the errors are drawn from independent standard normal distributions.

Figure 1: **Stimulated Critical Values for the LRM t-test (95th Percentile)**



Note: Critical values are computed via stochastic simulations using 40,000 replications for the LRM t-statistic in the Bewley instrumental variables regression:

$$y_t = \phi_0 - \phi_1 \Delta y_t + \psi_0 \boldsymbol{x}_t - \psi_1 \Delta \boldsymbol{x}_t + \mu_t$$

where $\psi_0 = -\frac{\pi_{yx}}{\pi_{yy}}$, the LRM, $\phi_0 = -\frac{c_0}{\pi_{yy}}$, $\phi_1 = -\frac{\pi_{yy}+1}{\pi_{yy}}$, $\psi_1 = \pi_{yx}$, and $\mu = -\frac{e}{\pi_{yy}}$ in the conditional ECM. A constant x_t, x_{t-1}, and y_{t-1} are used as instruments to estimate the model (Banerjee et al. 1993; De Boef and Keele 2008). The time series y_t and x_t are generated from: $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_t = \rho_x x_{t-1} + e_{xt}$ where the errors are drawn from independent standard normal distributions.